BIO 244: Unit 11

Some Key Results for Counting Process Martingales

This section develops some key results for martingale processes. We begin by considering the process $M(\cdot) \stackrel{\text{def}}{=} N(\cdot) - A(\cdot)$, where $N(\cdot)$ is the indicator process of whether an individual has been observed to fail, and $A(\cdot)$ is the compensator process introduced in the last unit. We show that $M(\cdot)$ is a zero mean martingale. Because it is constructed from a counting process, it is referred to as a counting process martingale. We then introduce the Doob-Meyer decomposition, an important theorem about the existence of compensator processes. We end by defining the quadratic variation process for a martingale, which is useful for describing its covariance function, and give a theorem that shows what this simplifies to when the compensator process is continuous.

Recall the definition of a martingale process:

Definition: The right-continuous stochastic processes $X(\cdot)$, with left-hand limits, is a Martingale w.r.t the filtration $(\mathcal{F}_t : t \geq 0)$ if it is adapted and

(a) $E \mid X(t) \mid < \infty \qquad \forall t$, and

(b) $E\left[X(t+s)|\mathcal{F}_t\right] \stackrel{a.s.}{=} X(t) \quad \forall s, t \geq 0.$

$X(\cdot)$ is a sub-martingale if above holds but with "=" in (b) replaced by "$\geq$"; called a super-martingale if "=" replaced by "$\leq$".

Let's discuss some aspects of this definition and its consequences:

- For the processes we will consider, the left hand limits of $X(\cdot)$ will always exist.

- Processes whose sample paths are a.s. right-continuous with left-hand limits are called **cadlag** processes, from the French *continu a droite, limite a gauche.*

- The condition that $E \mid X(t) \mid < \infty$ for every t is sometimes referred to as **integrability**. That is, $X(\cdot)$ is **integrable** if $E \mid X(t) \mid < \infty$ for every t. However, sometimes integrability is defined as the stronger condition that $sup_t E \mid X(t) \mid < \infty$.

- We later will use the property of **uniform integrability**, defined as $sup_t E\{\ |X(t)| \cdot 1[|X(t)| > c]\} \to 0$ as $c \to \infty$. Note: for X with $E|X| < \infty$, $|X|1_{\{|X| \leq c\}} \uparrow |X|$ as $c \to \infty$, so because of the monotone convergence theorem, $E\left(|X|1_{\{|X| \leq c\}}\right) \uparrow E\left(|X|\right)$, so $E\left(|X|1_{\{|X| > c\}}\right) = E\left(|X|\right) - E\left(|X|1_{\{|X| \leq c\}}\right) \to 0$ as $c \to \infty$. Uniform integrabillity: this holds uniformly in t, for $X(t)$. Also, uniform integrability implies integrability of each $X(t)$: $E|X(t)| \leq sup_t E(|X(t)|1_{|X(t)| > c}) + c$, which is finite for c large enough.

- We also can write (b) as

$$E[X(t+s) - X(t) \mid \mathcal{F}_t] \stackrel{a.s.}{=} 0.$$

- If $X(\cdot)$ is a martingale, $E[X(t)] = E[E[X(t) \mid \mathcal{F}_0]] = E[X(0)]$ because of the tower property of conditional expectations. For most of the examples we will consider, $X(0) = 0$, and hence $E[X(0)] = 0$. Note that we could have assumed this in the definition without loss of generality since if $X(\cdot)$ is a martingale with $E[X(0)] \neq 0$, $X^*(\cdot) = X(\cdot) - E[X(0)]$ is also a martingale, and has mean 0.

- When we introduced the definition of a martingale, we took

$$\mathcal{F}_t = \sigma(X(u) : 0 \leq u \leq t),$$

i.e., the (smallest) σ-algebra generated by $(X(u) : 0 \leq u \leq t)$. However, there can be other, richer, families to which $X(\cdot)$ is adapted and a martingale can be defined with respect to these.

For example, suppose that $X(t)$ denotes the change from baseline in a HIV patient's viral load t time units after randomization, and let Z denote gender. Consider $\{\mathcal{F}_t^* : t \geq 0\}$, where

$$\mathcal{F}_t^* = \sigma\left(X(u), 0 \leq u \leq t, Z\right),$$

and

$$\mathcal{F}_t = \sigma\left(X(u), 0 \leq u \leq t\right).$$

It follows that $\mathcal{F}_t \subset \mathcal{F}_t^*$, so that $\mathcal{F}^*$ is a "richer" filtration.

For $X(\cdot)$ to be a martingale with respect to $\mathcal{F}_t$, we would require

$$E[X(t+s)|\mathcal{F}_t] = E[X(t+s)|X(u), 0 \leq u \leq t] \overset{a.s.}{=} X(t).$$

For $X(\cdot)$ to be a martingale with respect to $\mathcal{F}^*$, however, we require

$$E[X(t+s)|\mathcal{F}_t^*] = E[X(t+s)|Z, X(u) : 0 \leq u \leq t] \overset{a.s.}{=} X(t).$$

$\Rightarrow$ One of these could hold and the other not hold. Thus, when we talk about a martingale, we must have a filtration in mind.

The Martingale M=N-A: The random walk example discussed in Unit 10 is one illustration of a Martingale. Now consider

$$N(t) \stackrel{def}{=} 1(U \leq t, \delta = 1)$$

and

$$A(t) \stackrel{def}{=} \int_0^t \lambda(u) Y(u)\, du,$$

where

$$Y(t) = 1(U \geq t).$$

We called $A(\cdot)$ the <u>compensator</u> for $N(\cdot)$ and showed that

$$E(A(t)) = E(N(t)) \quad \forall t.$$

Thus, $M(\cdot) \stackrel{def}{=} N(\cdot) - A(\cdot)$ is a zero-mean stochastic process.

Note that even though the process $Y(\cdot)$ is left continuous, the process $A(\cdot)$ is continuous for continuous T.

In the following, we will choose $\mathcal{F}_t = \sigma(N(u), N^C(u) : 0 \leq u \leq t)$, where

$$N^C(u) = 1(U \leq u, \delta = 0).$$

Notice: $\mathcal{F}_t = \sigma(N(u), Y(u) : 0 \leq u \leq t)$ is actually the same filtration.

Note that if λ is known and is > 0, knowing the history of $M(\cdot)$ up to time t is equivalent to knowing

$$\left(N(u), N^C(u) \quad : \quad 0 \leq u \leq t\right).$$

We now prove that $M(\cdot)$ is a martingale with respect to $\mathcal{F}_t$, following the proof in Fleming & Harrington, §1.3.

Proof: $M(\cdot)$ is right-continuous with left-hand limits. It is adapted since N is adapted and once $\mathcal{F}_t$ is known, Y is known until time t, so A is adapted as well. It's easy to show $E|M(t)| < \infty$ for any t. Thus we need to show that

$$E[M(t+s)|\mathcal{F}_t] \overset{a.s.}{=} M(t).$$

Since we can write $E[M(t+s)|\mathcal{F}_t]$ as

$$\begin{aligned} & E[N(t+s) - A(t+s)|\mathcal{F}_t] \\ = \ & N(t) - A(t) + E[N(t+s) - N(t)|\mathcal{F}_t] - E\left[\int_t^{t+s} \lambda(u)Y(u)du|\mathcal{F}_t\right], \end{aligned}$$

it is sufficient to show that

$$E[N(t+s) - N(t)|\mathcal{F}_t] \overset{a.s.}{=} E\left[\int_t^{t+s} \lambda(u)Y(u)du|\mathcal{F}_t\right]. \qquad (11.1)$$

If $Y(t) = 0$, both sides equal 0. If $Y(t) = 1$, $Y(t) = 1$ tells you all about $Y(v), N(v) : 0 \le v \le t$ since N is a (at most) 1-step process. Hence

$$\begin{aligned} & E\left[N(t+s) - N(t)|Y(v), N(v) : 0 \le v \le t\right] \\ = & \begin{cases} 0 & \text{if } Y(t) = 0 \\ E\left[N(t+s) - N(t)|Y(t) = 1\right] & \text{if } Y(t) = 1. \end{cases} \end{aligned}$$

Ditto for right-hand side of (11.1). Thus, we need to show that

$$E\left[N(t+s) - N(t)|Y(t) = 1\right] \overset{a.s.}{=} E\left[\int_t^{t+s} \lambda(u)Y(u)du|Y(t) = 1\right]. \qquad (11.2)$$

Since N(t+s)-N(t) is zero or one,

$$
\begin{aligned}
&E[N(t+s)-N(t)|Y(t)=1] \\
&\quad = P[N(t+s)-N(t)=1|Y(t)=1] \\
&\quad = P[t<U\leq t+s, \delta=1|U\geq t] \\
&\quad = \frac{P[t<U\leq t+s,\delta=1]}{P[U\geq t]}.
\end{aligned}
$$

Thus, in general, we can write

$$
E[N(t+s)-N(t)|Y(t)] = \frac{1[U\geq t]\cdot P[t<U\leq t+s,\delta=1]}{P[U\geq t]},
$$

where the last step follows because the event $\{Y(t)=1\}$ is the same event as $\{U\geq t\}$.

Now consider the right-hand side of (11.2). When $Y(t)=1$,

$$
\begin{aligned}
E\left[\int_t^{t+s}\ \lambda(u)Y(u)du|Y(t)=1\right] &= \int_t^{t+s}\lambda(u)E\left[Y(u)|Y(t)=1\right]du \\
= \int_t^{t+s}\lambda(u)P[U\geq u|U\geq t]du \\
= \frac{\int_t^{t+s}\lambda(u)P(U\geq u)du}{P(U\geq t)} = \frac{P[t<U\leq t+s,\delta=1]}{P[U\geq t]},
\end{aligned}
$$

where we have used the fact that

$$
\begin{aligned}
E[Y(u)|Y(t)=1] &= P[Y(u)=1|Y(t)=1] \\
&= P[U\geq u|U\geq t],
\end{aligned}
$$

and we have also used Exercise 1.

Thus, the right-hand side of (11.2) is expressible as

$$\frac{1(U \geq t) \cdot P[t < U \leq t+s, \delta = 1]}{P[U \geq t]},$$

and hence (11.2) holds, which means $M(\cdot) = N(\cdot) - A(\cdot)$ is a martingale.

Note: It can also be shown that this is a martingale when T is discrete (see F&H, §1.3).

We next consider an important result on the construction of martingales from submartingales. We first need to introduce the concept of a predictable process, which we do heuristically. See Fleming & Harrington or Andersen, Borgan, Gill & Keiding for a rigorous definition.

Definition: A stochastic process $X(\cdot)$ is called predictable (with respect to $\mathcal{F}_t$) if, for every t, $X(t)$ is measurable with respect to $\mathcal{F}_{t-}$ (its value is known just before time t). If $\mathcal{F}_t = \sigma(X(s) : s \leq t)$, that is, $X(t)$ is determined by

$$X(u), \quad 0 \leq u < t.$$

Example: T = continuous survival time. Then

$$A(t) \stackrel{def}{=} \int_0^t \lambda(u) Y(u) du$$

is predictable since its sample paths are continuous.

Example: Any left-continuous adapted process is predictable.

Example: $N(t)$, the number of failures at time t, is most of the time not predictable (unless the jumps can be completely predicted by the past).

We now turn to an important theorem whose proof is difficult and omitted (see Andersen, Borgan, Gill & Keiding for details). This version of the theorem comes from Fleming and Harrington, page 37:

Doob-Meyer Decomposition. Suppose $X(\cdot)$ is a non-negative submartingale adapted to $(\mathcal{F}_t : t \geq 0)$. Then there exists a right-continuous and nondecreasing predictable process $A(\cdot)$ such that $E(A(t)) < \infty$ for all t, and

$$M(\cdot) \overset{def}{=} X(\cdot) - A(\cdot)$$

is a martingale. If A(0)=0 a.s., then $A(\cdot)$ is unique.

- This theorem does not tell us what $A(\cdot)$ is, only that there exists some $A(\cdot)$ such that $M(\cdot) \overset{def}{=} X(\cdot) - A(\cdot)$ is a martingale.

- The process $A(\cdot)$ is called the compensator for $X(\cdot)$.
- We earlier noted that $N(t) = 1(U \leq t, \delta = 1)$ is a submartingale and showed that $N(t) - \int_0^t \lambda(u)Y(u)du$ is a martingale.

  This is an example of the theorem. Now we also know, however, that

$$\int_0^t \lambda(u)Y(u)du$$

  is the unique compensator for $N(t) = 1(U \leq t, \delta = 1)$.

Next consider the covariance function for a martingale process. Suppose $M(\cdot)$ is some martingale for which $M(0) = 0$ and that $E(M^2(t)) < \infty$ for all t. Then one can show (compare with Exercise 6 of Unit 10) that for any $s \geq 0$ and $t \geq 0$,

$$\text{Cov}(M(t+s), M(t)) = \text{Var}M(t).$$

Thus, we know the covariance function of a zero-mean martingale by knowing $E\left(M^2(t)\right)$. It is straightforward to show that this implies that $M(\cdot)$ has uncorrelated increments.

Note, however, that the process $M^2(\cdot)$ is right-continuous and suppose that $E \mid M^2(t) \mid = EM^2(t) < \infty$. From Jensen's inequality ("if you average first and then take convex function you get something smaller", see Billingsley, Probability and Measure 1986, equation (34.7)),

$$\begin{aligned} E\left[M^2(s+t) \mid \mathcal{F}_t\right] &\geq \left(E\left[M(s+t) \mid \mathcal{F}_t\right]\right)^2 \\ &\overset{a.s.}{=} M^2(t). \end{aligned}$$

Thus, $M^2(\cdot)$ is a submartingale. By the Doob-Meyer decomposition, there exists a unique predictable process, which we will denote by $< M, M > (\cdot)$, such that $M^2(\cdot) - < M, M > (\cdot)$ is a martingale.

- The compensator, $< M, M > (\cdot)$, for $M^2(\cdot)$ is called the **predictable quadratic variation** process corresponding to $M(\cdot)$.
- $$E(M^2(t) - < M, M > (t)) = 0 \quad \Rightarrow \quad E(M^2(t)) = E(< M, M > (t)).$$

  Thus, if we could find $< M, M > (t)$, we could take its expectation to find $E(M^2(t))$ and thus $\text{Var}(M(t))$.

- Consider the "special" martingale

$$M(t) = N(t) - A(t),$$

where $N(t) = 1(U \leq t, \delta = 1)$ and $A(t) = \int_0^t \lambda(u)Y(u)du$. We will see below that

$$< M, M > (\cdot) = A(\cdot).$$

Thus, for <u>this</u> martingale

$$\begin{aligned} Var(M(t)) &= E(M^2(t)) = E(< M, M > (t)) = E(A(t)) \\ &= \int_0^t \lambda(u)E(Y(u))du = \int_0^t \lambda(u)P(U \geq u)du. \end{aligned}$$

Theorem (see Theorem 2.5.1 in F&H, p. 74): Suppose $N(\cdot)$ is a counting process with continuous compensator $A(\cdot)$, and that the resulting martingale $M(\cdot) = N(\cdot) - A(\cdot)$ satisfies $EM^2(t) < \infty$ for all t. Then

$$< M, M > (\cdot) \stackrel{a.s.}{=} A(\cdot).$$

This is a very useful result because it allows us to find the variance of a counting process martingale as the expectation of its compensator. We omit the full proof, which requires a somewhat delicate treatment of Lebesgue-Stieltjes integrals. It involves expressing $M^2(t)$ as

$$2 \int_0^t M(s-)dM(s) + \sum_{s \leq t} [\Delta M(s)]^2,$$

where the term on the right represents the jump points of $M(\cdot)$. Since these jumps are all of size one, this term equals N(t), so that one can write

$$M^2(t) - A(t) = 2 \int_0^t M(s-)dM(s) + M(t).$$

It is then shown that the right-hand side of the above equation is a martingale, which implies that $A(\cdot)$ is the compensator for $M^2(\cdot)$. Since A(0)=0, it follows that this compensator is unique; that is, that $A(\cdot) \stackrel{a.s.}{=} < M, M > (\cdot)$.

Example: Suppose that $T \sim Exp(\lambda)$ and that the c.d.f. of the censoring variable C is denoted $G(\cdot)$. Then

$$A(t) = \int_0^t \lambda Y(u)du,$$

so that

$$\begin{aligned} E(A(t)) &= \lambda \int_0^t E(Y(u))du \\ &= \lambda \int_0^t P(T \geq u)P(C \geq u)du \\ &= \lambda \int_0^t e^{-\lambda u}(1 - G(u))du. \end{aligned}$$

Thus, the zero mean martingale process $M(\cdot) = N(\cdot) - A(\cdot)$ has covariance function:

$$Cov(M(t), M(t+s)) = Var(M(t)) = \lambda \int_0^t e^{-\lambda u}(1 - G(u))du.$$

Exercises

1. Prove $\int_t^{t+s} \lambda(u)P(U \geq u)du = P[t < U \leq t+s, \delta = 1]$ (see page 8).

2. Prove that if a, b are any constants and $M_1(\cdot)$ and $M_2(\cdot)$ are martingales with respect to the same filtration $\{\mathcal{F}_t : t \geq 0\}$,

$$E\left[a \cdot M_1(t+s) + b \cdot M_2(t+s) \mid \mathcal{F}_t\right] = a \cdot M_1(t) + bM_2(t).$$

   Use this to show

   (a) $a \cdot M_1(\cdot)$ is a martingale,

   (b) $M_1(\cdot) + M_2(\cdot)$ is a martingale; and

   (c) $M_1(\cdot) - M_2(\cdot)$ is a martinagale.

3. Let $N(\cdot)$ be the counting process defined by $N(t) = 1(U \leq t, \delta = 1)$, and let $A(\cdot)$ be its compensator, where $A(t) = \int_0^t \lambda(s)Y(s)ds$. Assuming that survival time T is a continuous random variable, show that the compensator process $A(\cdot)$ is continuous.

4. Suppose that $W(\cdot)$ is a Wiener process.

   (a) Define a Wiener process and prove that it has independent increments.

   (b) Is $W(\cdot)$ a martingale? Justify your answer. You may want to use Theorem 34.1 from Billingsley, Probability and Measure, 1986, after a careful choice of π-system.

5. Show that $E|M(t)| < \infty$ on page 6 (in fact, it is even less than or equal to 2).