# BIO 244: Unit 13

## Martingale Central Limit Theorem and Related Results

In this unit we discuss a version of the martingale central limit theorem, which states that under certain conditions, a sum of orthogonal martingales converges weakly to a zero-mean Gaussian process with independent increments. In subsequent units we will use this key result to find the asymptotic behavior of estimators and tests under a variety of conditions. Many of the results we present hold under more general conditions, and most are presented without proof. See the texts by Fleming & Harrington and by Andersen, Borgan, Gill & Keiding for details. We begin with some preliminary results.

In what follows, suppose that:

$N(\cdot)$ is a counting process.

$A(\cdot)$ is the compensator for $N(\cdot)$ (assume A$(\cdot)$ is continuous).

$M(\cdot) \stackrel{def}{=} N(\cdot) - A(\cdot)$ is a zero-mean martingale.

Recall from Unit 12 that

- If $H(\cdot)$ is a bounded and predictable process, defined on the same filtration as $M(\cdot)$, then the process $Q(\cdot)$ defined at time $t$ by

  $$Q(t) \stackrel{def}{=} \int_0^t H(s)dM(s)$$

  is a zero-mean martingale.

- If in addition $E(M^2(t)) < \infty$, then

$$< Q, Q > \ (t) \overset{a.s.}{=} \int_0^t H^2(s) dA(s),$$

and thus

$$Var(Q(t)) = E(Q^2(t)) = E(< Q, Q > (t)) = E(\int_0^t H^2(s) dA(s)).$$

Recall that some of the commonly-used statistics for analyzing survival data can be expressed as functions of stochastic integrals with respect to counting process martingales. The above result helps to determine the variance of these statistics.

- If the counting processes $N_1(\cdot)$, $N_2(\cdot), \cdots$ are defined on the same filtration, have continuous compensators, and no 2 jump at the same time, then the corresponding martingales $M_1(\cdot)$, $M_2(\cdot), \cdots$ are orthogonal. That is, $< M_i, M_j > (\cdot) \overset{a.s.}{=} 0$ for $i \neq j$.

In preceding units we have ignored the so-called "usual conditions" ("les conditions habituelles") for filtrations. These are completeness and right continuity. First, completeness. Completeness of a filtration $\mathcal{F}_t$ means that $\mathcal{F}_0$ contains all null sets (a null set is a set $A$ such that there exists a set $B$ with $A \subset B$, $B \in \mathcal{F}$, and $P(B) = 0$). The collection of null sets is usually denoted by $\mathcal{N}$. Next, right continuity. Right continuity of a filtation means that $\mathcal{F}_t = \cap_{\epsilon>0}\mathcal{F}_{t+\epsilon}$. Intuitively that means that you can look a little bit into the information in the future, but only for an infinitesimal amount of time.

The usual conditions for filtrations are sometimes necessary for theorems to hold. For example, right-continuity is needed for the Doob-Meyer decomposition. Some books (e.g. Andersen et al.) implicitly always assume that the usual conditions hold. Also, right continuity of a filtration can be a powerful tool to prove that something is a stopping time. As we will see later in this unit, this has implications for so-called localizing sequences. Consider events of the form $\tau = inf_t\{\mid (X(t) \mid > k\}$. We sometimes want such events to be stopping times. To prove this for particular processes, we will use right-continuity of the filtration.

An example of a right-continuous filtration is a filtration generated by a right-continuous jump process $K$ (De Sam Lazaro, 1974, Lemma 3.3). $K$ is a right-continuous jump process if for all $t$, $\omega$, $K(s,\omega)$ is constant in some time interval $s \in [t, t+\epsilon)$ for some $\epsilon > 0$ ($\epsilon$ possibly depending on $t, \omega$).

If a filtration does not include all null sets, one can go over to the smallest filtration that includes both the original filtration and all null sets. This is called "completion" of the filtration. Completion of a filtration preserves right-continuity.

If for some filtration $\mathcal{F}_t$ the usual conditions do not hold, one can go over to the so-called augmented filtration $\mathcal{F}_t^a$, which is the smallest filtration containing $\mathcal{F}_t$ for which the usual conditions do hold. One can show that such filtration exists, and that (Rogers and Williams, 1994, Lemma 67.4)

$$\mathcal{F}_t^a = \cap_{s>t}\sigma\left(\mathcal{F}_s, \mathcal{N}\right) = \sigma\left(\cap_{s>t}\mathcal{F}_s, \mathcal{N}\right).$$

If $M$ is a martingale with respect to some filtration $\mathcal{F}_t$, then it is also a martingale with respect to $\mathcal{F}_t^a$ (Lemma 67.10 from Rogers and Williams, 1994). In many cases this makes it possible to ignore these "usual conditions."

Suppose that $< M_i, M_j > (\cdot)$ and $< Q_i, Q_j > (\cdot)$ denote the compensators for $M_i(\cdot)M_j(\cdot)$ and $Q_i(\cdot)Q_j(\cdot)$. A useful result is the following (see F & H, Thm 2.4.2):

**Theorem 13.1:** Suppose that $N_1(\cdot), N_2(\cdot), \ldots$ are bounded counting processes ($N_i$ bounded by $K_i$), $M_1(\cdot), M_2(\cdot), \cdots$ are the corresponding zero-mean counting process martingales, each $M_i$ satisfies $EM_i^2(t) < \infty$ for any $t$, and that $H_1(\cdot), H_2(\cdot), \cdots$ are bounded and predictable processes. Suppose also that the filtration concerned is right-continuous. Let

$$Q_i(t) \stackrel{def}{=} \int_0^t H_i(u) dM_i(u) \qquad i = 1, 2, \cdots .$$

Then $< Q_i, Q_j > (t) \stackrel{a.s.}{=} \int_0^t H_i(s) H_j(s) d < M_i, M_j > (s)$.

**Note:** Since the logrank numerator and Cox's score function can be expressed as sums of this type of stochastic integral, we can use this result to find their variances. See exercises.

**Corollary 13.1:** If $M_1(\cdot), M_2(\cdot), \cdots$ are orthogonal, then so are $Q_1(\cdot), Q_2(\cdot), \cdots$.

To see the value of this, let $M_1(\cdot), M_2(\cdot), \ldots$ be orthogonal, square integrable counting process martingales arising from bounded counting processes. Define

$$\Sigma Q(t) \stackrel{def}{=} \sum_{i=1}^{n} Q_i(t). \tag{13.1}$$

Then

$$\begin{aligned} Var(\Sigma Q(t)) &= E\left((\Sigma Q(t))^2\right) = E(Q_1(t) + \cdots + Q_n(t))^2 = E\Big(\sum_{i,j=1}^{n} Q_i(t) Q_j(t)\Big) \\ &= E\Big(\sum_{i,j=1}^{n} < Q_i, Q_j > (t)\Big) = E\Big(\sum_{i=1}^{n} Q_i(t)^2\Big). \end{aligned}$$

Thus, the variance function of $\Sigma Q(\cdot)$ is given by

$$Var(\Sigma Q(t)) = \sum_{i=1}^{n} E\left(\int_0^t H_i^2(s)\ dA_i(s)\right). \tag{13.2}$$

This is useful for finding the variance of statistics of the form in (13.1). Note that the theorem is silent about the correlation between $H_1(\cdot), \cdots, H_n(\cdot)$. As we shall see, these processes are correlated in some applications of this theorem.

We next consider 'local' properties, which allow us to relax conditions needed for the Martingale central limit theorem. For details, see F & H, Ch. 2, or A,B,G & K.

A deterministic function $f(t)$, $0 \leq t < \infty$, is said to have a property locally if the property holds on $[0, s]$ for every $s$. For example, $f(\cdot)$ is "locally bounded" if, for every $s$, there is a constant $c_s$ such that

$$\sup_{0\leq u\leq s} \mid f(u) \mid \leq c_s\ .$$

**Note:** Being locally bounded is a weaker condition than being bounded. For example, the function $f(t) = t$ is locally bounded but not bounded. The idea of "locally" can also be defined by the existence of a sequence of constants, say $t_1, t_2, \cdots$, satisfying $t_s \to \infty$ as $s \to \infty$, such that the condition holds on $[0, t_s]$ for every $s$. For example, $f(\cdot)$ is locally bounded if there exists $t_1, t_2, \cdots$ and $c_1, c_2, \cdots$ such that $t_s \to \infty$ and $\sup_{0\leq u\leq t_s} \mid f(u) \mid \leq c_s$ , $s = 1, 2, \cdots$.

The same ideas apply to stochastic processes, with the only change being that the sequence of times consists of random variables defined on the same filtration as the process.

**Definition:** An increasing sequence of random variables $\tau_1, \tau_2, \ldots$ is a localizing sequence with respect to a filtration $(\mathcal{F}_t : t \geq 0)$ if for each $n$,

$\{\tau_n \leq t\} \in \mathcal{F}_t$ for all $t \geq 0$, and if $lim_{n\to\infty}\tau_n = \infty$ a.s.

Any particular $\tau_n$ is called a "stopping time".

**Definition** A process $X(\cdot)$ is locally bounded if there exists a localizing sequence $(\tau_n)$ and constants $(c_n)$ such that for each n,

$$\sup_{0\leq t\leq\tau_n} \mid X(t) \mid \overset{a.s.}{\leq} c_n.$$

**Lemma 13.1** Any adapted cadlag process (right-continuous with left-hand limits) $X(\cdot)$ adapted to a right-continuous filtration, satisfying $X(0)$ is bounded by say $K_1$ and having jump sizes bounded by say $K_2$ is locally bounded.

Proof: this can be seen by taking

$$\tau_n = \sup\{t : |X(s)| \leq n : 0 \leq s \leq t\} \wedge n \vee 0.$$

Then for $t \geq n$, $\{\tau_n \leq t\} = \Omega \in \mathcal{F}_t$, and for $t < n$,

$$\{\tau_n < t\} = \cup_{s<t, s\in\mathbb{Q}} \{|X(s)| > n\} \in \mathcal{F}_t.$$

Hence for every $\delta > 0$,

$$\{\tau_n \leq t\} = \cap_{\epsilon\in\mathbb{Q},\epsilon\in(0,\delta]} \{\tau_n < t + \epsilon\} \in \mathcal{F}_{t+\delta},$$

since each of the events in the intersection is in $\mathcal{F}_{t+\epsilon} \subset \mathcal{F}_{t+\delta}$. Hence, by right continuity of the filtration, $\{\tau_n \leq t\} \in \mathcal{F}_t$. Check for yourself that $\tau_n \to \infty$ a.s., and that $X(t \wedge \tau_n)$ is bounded:

$$|X(t \wedge \tau_n)| \leq K_1 \vee (n + K_2).$$

**Corollary 13.2:** Any counting process martingale $M(\cdot)$ on a right-continuous filtration and satisfying $M(0) \overset{a.s.}{=} 0$ is locally bounded if the compensator of the counting process is continuous.

**Definition:** For any stochastic process $X(\cdot)$ and localizing sequence $(\tau_n)$, the stopped process $X_n(\cdot)$ is defined by $X_n(t) = X(t\wedge\tau_n)$. That is, $X_n(t) = X(t)$

for $t \leq \tau_n$ and equal to $X(\tau_n)$ for $t > \tau_n$.

For example, another way to define a "locally bounded" stochastic process $X(\cdot)$ is by requiring that there exists a stopped process $X_n(\cdot)$ that is bounded for each n.

It turns out that stopping a martingale preserves the martingale property (Optional Stopping Theorem, Theorem 2.2.2 in Fleming and Harrington). Hence, if a martingale $M$ is locally bounded, there exists a localizing sequence $\tau_n$ such that the stopped process is a bounded martingale for each $n$.

Next, recall Wiener Processes (or Brownian Motion)

**Definition:** If $W(\cdot)$ is a Gaussian process satisfying $W(0) = 0$, $E(W(t)) = 0$ for all t, and $\text{Cov}(W(s), W(t)) = \min(s,t)$ for all t,s, then $W(\cdot)$ is a Wiener process.

**Theorem 13.2:** Suppose $W(\cdot)$ is a Wiener process and $f(\cdot)$ is any (deterministic) nonnegative function which is bounded on every bounded interval (locally bounded). Then

(1) $W(\cdot)$ is a zero-mean martingale.

(2) The predictable quadratic variation process for $W(\cdot)$ satisfies

$$< W, W > (t) = t.$$

(3) $Q(t) \stackrel{def}{=} \int_0^t f(s)dW(s)$ is a zero-mean Gaussian process with Q(0)=0, independent increments, and variance function

$$\text{var}\ (Q(t)) = \int_0^t f^2(s)ds.$$

**Proof:** (1) is Exercise 4b of Unit 11. To prove (2), we need to show that, for every $t \geq 0$ and $s \geq 0$

$$E[W^2(t+s) - (t+s) \mid \mathcal{F}_t] \stackrel{a.s.}{=} W^2(t) - t,$$

or that

$$E[W^2(t+s) \mid \mathcal{F}_t] \stackrel{a.s.}{=} W^2(t) + s.$$

Note that we can write

$$\begin{aligned}
E[W^2(t+s) \mid \mathcal{F}_t] &= E[(W(t+s) - W(t))^2 - W^2(t) + 2W(t+s)W(t) \mid \mathcal{F}_t] \\
&= E[(W(t+s) - W(t))^2 \mid \mathcal{F}_t] - W^2(t) + 2E[W(t+s)W(t) \mid \mathcal{F}_t] \\
&= E[(W(t+s) - W(t))^2 \mid \mathcal{F}_t] - W^2(t) + 2W(t)E[W(t+s) \mid \mathcal{F}_t] \\
&= E[(W(t+s) - W(t))^2 \mid \mathcal{F}_t] + W^2(t),
\end{aligned}$$

where we have used the martingale property $E[W(t+s) \mid \mathcal{F}_t] = W(t)$. Thus, the result follows from the fact that $E[(W(t+s) - W(t))^2 | \mathcal{F}_t] = E[(W(t+s) - W(t))^2] = var(W(t+s)) + var(W(t)) - 2cov(W(t+s), W(t)) = (t+s) + t - 2t = s$. The first equality here can be seen as follows. If $X$ is independent of $(Y_1, Y_2, \ldots, Y_k)$, then $E[X|Y_1, \ldots, Y_k] = EX$. Intuitively, this is clear: knowledge of $(Y_1, Y_2, \ldots, Y_k)$ does not predict the expectation of $X$. The proof for infinitely many $Y$'s can be done using Theorem 34.1 from Billingsley, Probability and Measure, 1986, after a careful choice of $\pi$-system, following the same lines as Exercise 4b of Unit 11. See Exercises.

(3) follows from a more general result that, subject to some conditions, integrals of bounded and predictable functions with respect to local martingales are themselves local martingales. See F&H, Lemma 2.4.1 for details. Note that the martingale does not need to be a counting process martingale.

Finally, for any $\epsilon > 0$ and stochastic integral $U(\cdot)$ of the form

$$U(t) = \int_0^t H(s)dM(s),$$

where $M(\cdot)$ is a counting process martingale with continuous compensator, define the process $U_\epsilon(\cdot)$ by

$$U_\epsilon(t) = \int_0^t H(s)1[\mid H(s) \mid \geq \epsilon]dM(s).$$

Note that a jump in the process $U(\cdot)$, say at time $s$, has magnitude $H(s)$. Hence, $U_\epsilon(\cdot)$ contains only those jumps in $U(\cdot)$ that are larger than $\epsilon$.

We now are ready to state a version of the Martingale Central Limit Theorem. Suppose the filtration concerned is right-continuous. For any $n$ and $i = 1, 2, \cdots, n$, suppose

- $N_{in}(\cdot)$ is a counting process with continuous compensator $A_{in}(\cdot)$ (13.3)
- $H_{in}$ is locally bounded and predictable, and (13.4)
- No two of the counting processes can jump at the same time, so that the $n$ martingales $M_{in}(\cdot) = N_{in}(\cdot) - A_{in}(\cdot)$ are orthogonal. (13.5)

Define

- $U_{in}(t) \stackrel{def}{=} \int_0^t H_{in}(s) dM_{in}(s)$
- $\Sigma U_n(t) \stackrel{def}{=} \sum_{i=1}^n U_{in}(t)$
- $U_{in,\epsilon}(t) \stackrel{def}{=} \int_0^t H_{in}(s) 1[|H_{in}(s)| \geq \epsilon] dM_{in}(s)$ , and
- $\Sigma U_{n,\epsilon}(t) \stackrel{def}{=} \sum_{i=1}^n U_{in,\epsilon}(t)$.

**Martingale Central Limit Theorem**. Assume (13.3)-(13.5), and that for every $t$,

(a) $< \Sigma U_n, \Sigma U_n > (t) \stackrel{p}{\to} \alpha(t)$ (some deterministic function)

and

(b) $< \Sigma U_{n,\epsilon}, \Sigma U_{n,\epsilon} > (t) \stackrel{p}{\to} 0 \quad \forall \epsilon > 0$

as $n \to \infty$, then as $n \to \infty$

$$\Sigma U_n(\cdot) \stackrel{w}{\to} U(\cdot),$$

where $U(\cdot)$ is a zero-mean Gaussian process with independent increments and variance function $\alpha(\cdot)$.

Note:

- The proof of this theorem can be found in Fleming and Harrington Chapter 5.
- Condition (b) ensures that as $n$ gets large, there cannot be too many big jumps in $\Sigma U_n(\cdot)$ (since if there were, this process might not converge to zero). Thus, (b) can be viewed as a "tightness" condition.
- We can write $U(\cdot)$ as $U(\cdot) \stackrel{def}{=} \int_0^{\cdot} f(s)dW(s)$, where $W(\cdot)$ is a Wiener process, and where $f(\cdot)$ is such that $\int_0^t f^2(s)ds = \alpha(t)$ (Theorem 13.2). From this it follows that

$$\text{var }(U(t)) = \int_0^t f^2(s)ds = \alpha(t).$$

- Strictly speaking, all condition (a) says is that $< \Sigma U_n, \Sigma U_n > (t)$ converges in probability to some determistic function; it doesn't require that this limit equals some specific value. In practice, this is usually proven by actually finding the probability limit of $< \Sigma U_n, \Sigma U_n > (t)$. This is useful because the limiting function (denoted $\alpha(\cdot)$) also represents the variance function of the limiting Gaussian process.
- Finding the limit in probability of $< \Sigma U_n, \Sigma U_n > (t)$ is facilitated by the fact that (compare with Theorem 13.1; local boundedness of $H_{in}$ is enough, see Fleming and Harrington Theorem 2.4.3)

$$< \Sigma U_n, \Sigma U_n > (t) = \sum_{i=1}^{n} \int_0^t H_{in}^2(s)dA_{in}(s).$$

  Similarly, finding the limit in probability of $< \Sigma U_{n,\epsilon}, \Sigma U_{n,\epsilon} > (t)$ is facilitated by the fact that

$$< \Sigma U_{n,\epsilon}, \Sigma U_{n,\epsilon} > (t) = \sum_{i=1}^{n} \int_0^t \left(H_{in}^2(s)\right) \cdot 1 \left[H_{in}(s) \geq \epsilon\right] dA_{in}(s).$$

- Note that the counting processes $N_{in}(\cdot)$ can be any counting processes, not just the simple 'survival' counting process that can jump at most once.

- In the definition of $\Sigma U_n(\cdot)$, one might have expected to see the multiplier $n^{-1/2}$, analogous to the ordinary CLT. As we will see later, this is implicitly contained in the integrands $H_{in}(\cdot)$.

## Additional Details

The conditions for the Martingale CLT apply more generally than stated earlier in this unit. As seen below, one way is by relaxing the condition on the martingales $M_{in}(\cdot)$ to be local martingales, as defined below.

- **Definition:** A process $M(\cdot)$ is a local martingale (or sub-martingale) if there exists a localizing sequence $(\tau_n)$ such that the stopped process $M_n(\cdot)$ is a martingale (sub-martingale) for every $n$.

  **Note 1:** Consider the stopped process $M_n(\cdot)$. For any fixed $t$, $M_n(t)$ equals $M(t)$ or $M(\tau_n)$, depending on whether the random variable $\tau_n$ is $> t$ or $\leq t$, respectively. Similarly, $M_n(t+s)$ equals $M(t+s)$ or $M(\tau_n)$, depending on whether the random variable $\tau_n$ is $> t+s$ or $\leq t+s$, respectively. Thus, the change in $M_n(\cdot)$ between $t$ and $t+s$ equals 0, $M(\tau_n) - M(t)$, or $M(t+s) - M(t)$, depending on whether $\tau_n < t$, $t \leq \tau_n < t+s$, or $\tau_n \geq t+s$, respectively. In all cases, this stopped process will have uncorrelated increments.

  **Note 2:** Any martingale is a local martingale.

**Definition:** A local martingale $M(\cdot)$ is called square integrable if $sup_t E(M(t)^2) < \infty$.

**Theorem:** Let $N(\cdot)$ be a counting process with continuous compensator $A(\cdot)$ on a right-continuous filtration. If $A(\cdot)$ is locally bounded, then $M(\cdot) = N(\cdot) - A(\cdot)$ is a local square integrable martingale.

The proof is easy. Because of Lemma 13.1, $N$ is locally bounded. Hence also the sum $N - A$ is locally bounded (why?). But then it surely is locally square integrable. It is a martingale because of the definition of compensator. Because of the Optional Stopping Theorem on page 7, it hence is a locally square integrable martingale.

**Extended Doob-Meyer Decomposition:** If $X(\cdot)$ is a nonnegative local submartingale on a right-continuous filtration, there exists a right-continuous, nondecreasing, predictable process $A(\cdot)$ such that $X(\cdot) - A(\cdot)$ is a local martingale. Further, if $A(0) = 0$ a.s., then $A(\cdot)$ is unique.

**Theorem:** Suppose that $H(\cdot)$ is a locally bounded and predictable process and $M(\cdot) = N(\cdot) - A(\cdot)$ is a local counting process martingale with continuous compensator and $E(M(t)^2) < \infty$. Suppose the filtration is right-continuous. Then:

$$Q(t) = \int_0^t H(s)dM(s) \text{ is a local square integrable martingale, and}$$

$$< Q, Q > (t) \overset{a.s.}{=} \int_0^t H^2(s)dA(t) \text{ for all t.} \tag{13.3}$$

For a proof, see F & H, Theorem 2.4.3. The statement can be found in ABGK, Theorem II.3.1. The value of this Theorem is that we no longer require the integrand $H(\cdot)$ to be bounded, and we still get a nice result to help in finding the variance function of $Q(\cdot)$.

Exercises

1. Suppose $N_1(\cdot)$ and $N_2(\cdot)$ are counting processes that are defined on the same filtration and that cannot jump at the same time. Let $A_i(\cdot)$ be the compensator processes for $N_i(\cdot)$, and assume that $A_i(\cdot)$ is continuous, i=1,2. Let $M_i(\cdot) = N_i(\cdot) - A_i(\cdot)$, i=1,2 and assume that $H(\cdot)$ is a bounded and predictable process. Define $Q_i(t) = \int_0^t H(s)dM_i(s)$, for i=1,2.

   (a) Define a counting process and prove that every counting process is a submartingale.

   (b) Is $A_1(\cdot) - M_1^2(\cdot)$ a martingale? Justify your answer.

   (c) Is $Q_1(\cdot)Q_2(\cdot)$ a martingale? Justify your answer.

   (d) Is $Q_1(\cdot)Q_1(\cdot)$ a martingale? Justify your answer.

2. Complete the proof of Theorem 13.2, (2).

3. (15 points). We expressed the numerator of the logrank test and the score of the Cox proportional hazards model as

$$\sum_{i=1}^{n} \int_0^\infty H_{in}(s)dM_i(s),$$

   with $H_{in}$ predictable processes and $M_i$ martingales. Next we want to apply the martingale Central Limit Theorem to

$$\sum_{i=1}^{n} \int_0^t H_{in}(s)dM_i(s).$$

   Suppose we are interested in the limiting distribution of this process at a fixed time $t$. Can we use the ordinary Central Limit Theorem for random variables with values in $\mathbb{R}^k$? Please explain.

4. (30 points). Suppose $T_i$ and $C_i$ are survival- and censoring times, and $Z_i$ the corresponding patient's covariate (suppose a scalar covariate). Suppose that we observe $U_i = \min(C_i, T_i)$, $\delta_i = 1_{T_i \le C_i}$, and $Z_i$ $(i = 1, \ldots, n)$. Suppose that $(T_i, C_i, Z_i)$ $(i = 1, \ldots, n)$ are iid. Suppose that the hazard of $T_i$ follows the Cox proportional hazards model:

$$\lambda_i(t) = \lambda_0(t) e^{\beta Z}.$$

As usual, define $N_i(\cdot)$ as the counting process which jumps when patient $i$ is observed to fail, and $M_i$ the corresponding counting process martingale. Define $Y_i(t)$ as the indicator for whether patient $i$ is at risk at time $t$.

(a) (5 points). Give an expression for $M_i$ in terms of the hazard and the observables.

(b) (5 points). Define

$$H_{in}(t) = \frac{Y_i(t) e^{\beta Z_i}}{\sqrt{\sum_{j=1}^{n} Y_j(t) e^{\beta Z_j}}}.$$

Find a condition under which $H_{in}$ is bounded for each $n$.

(c) (20 points). Consider

$$X_n(t) = \sum_{i=1}^{n} \int_0^t \frac{Y_i(s) e^{\beta Z_i}}{\sqrt{\sum_{j=1}^{n} Y_j(s) e^{\beta Z_j}}} dM_i(s).$$

Show that $X_n(\cdot)$ converges weakly. You may interchange convergence in probability and taking integrals if needed (we will see in Unit 15 that that is valid here). Indicate all things you checked, even if they look trivial. What is the limiting distribution of $X_n(\cdot)$?

## References

Rogers L C G and D Williams (1994). *Diffusions, Markov processes, and martingales.* John Wiley and Sons, Chichester, New York, Wiley series in probability and mathematical statistics, volume 1.

De Sam Lazaro, J (1974). *Sur les helices du flot special sous une fonction.* Zeitschrift fuer Wahrscheinlichkeitstheorie und verwandte Gebiete 30, Springer–Verlag, Berlin, pp 279–302.